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Multifractal Nonrigidity of Topological Markov Chains

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Abstract Given a multifractal spectrum, we consider the problem of whether it is possible to recover the potential that originates the spectrum. The affirmative solution of this problem would correspond to a "multifractal" classification of dynamical systems, i.e., a classification solely based on the information given by multifractal spectra. For the entropy spectrum on topological Markov chains we show that it is possible to have both multifractal rigidity and multifractal "nonrigidity", by appropriately varying the Markov chain and the potential defining the spectrum. The "nonrigidity" even occurs in some generic sense. This strongly contrasts to the usual opinion among some experts that it should be possible to recover the potential up to some equivalence relation, at least in some generic sense.

Keywords Multifractal analysis · Multifractal rigidity

1 Introduction

Roughly speaking, the theory of multifractal analysis studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. For example, we can consider Birkhoff averages, Lyapunov exponents, pointwise dimensions, or local entropies. Since the level sets of these quantities are rarely manifolds, in order to measure their complexity it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension. This gives rise to several multifractal spectra, such as entropy spectra and dimension spectra. On the other hand, the theory of multifractal analysis is closely related to the experimental study of dynamical systems. In particular, the multifractal spectra can be determined experimentally with an arbitrary precision. Furthermore, the multifractal spectra

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contain information about the dynamical system that may perhaps be used to recover the dynamics, probably not completely but at least in some equivalence class. We refer to this question as a *multifractal rigidity* problem. The theory of multifractal rigidity asks in particular whether it is possible to use the multifractal spectra to restore the dynamics, again perhaps up to some equivalence relation, such as cohomology or conjugation by automorphisms. We refer to [4] for a related discussion.

Our main objective is to show that, for the model class of topological Markov chains, both the phenomena of multifractal *rigidity* and multifractal "*nonrigidity*" do occur. One of the main problems of multifractal rigidity is how to effectively recover the local characteristics of the system, if possible providing some appropriate algorithm. There was some hope that this could perhaps be effected for a large class of multifractal spectra and several model classes of hyperbolic dynamical systems. This was supported by former work in [1, 2, 5]. Nevertheless, we will give explicit examples of topological Markov chains for which there is no multifractal rigidity, even in some generic sense.

In order to briefly describe our results rigorously we recall the notion of entropy spectrum (see Sect. 2 for details). We consider topologically mixing one-sided Markov chains (Σ_A^+, σ), associated to a $p \times p$ transition matrix A with $\Sigma_A^+ \subset \{1, \ldots, p\}^{\mathbb{N}}$, where σ is the shift map. Given a function $\varphi \colon \Sigma_A^+ \to \mathbb{R}$, the *entropy spectrum* of φ is defined by

$$\mathcal{E}(\alpha) = h\left(\sigma \mid \left\{x \in \Sigma_A^+ : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\sigma^k x) = \alpha\right\}\right),$$

where $h(f|Z)$ is the topological entropy of f on Z (notice that Z need not be compact; see for example [4] for the definition). For a class of hyperbolic dynamical systems and a class of sufficiently regular functions (such as Hölder continuous functions) it is possible to show that:

1. the domain of $\mathcal{E}$ is an interval that may reduce to a single point;
2. $\mathcal{E}$ is either a delta function, or is analytic and strictly convex (this last alternative occurs if and only if φ is not cohomologous to a constant, in which case the domain is not a single point; see (7) below).

We refer to [4] for details and references. The main objective of the theory of multifractal rigidity is to recover as much information as possible about the function φ from the entropy spectrum $\mathcal{E}$, at least up to some equivalence relation. We will say in this paper that two functions φ and ψ are *equivalent* if

$$\varphi \circ \tau - \psi = u \circ \sigma - u + c \tag{1}$$

for some automorphism $\tau \colon \Sigma_A^+ \to \Sigma_A^+$, continuous function $u \colon \Sigma_A^+ \to \mathbb{R}$, and constant $c \in \mathbb{R}$ (see Sect. 2.2 for a detailed discussion).

We can now briefly describe our results. Namely, with respect to the equivalence relation in (1) we show the following:

1. For a function φ constant on cylinders of length 2 in $\Sigma_A^+ \subset \{1, 2\}^{\mathbb{N}}$ there is a strong multifractal rigidity. Namely, we show that it is possible to completely characterize the equivalence class of φ from the entropy spectrum, except from a particular spectrum for the full topological Markov chain (in which case there are three possible equivalence classes). See Sect. 4.2.
2. For a function φ constant on cylinders of length 2 in $\Sigma_A^+ \subset \{1, 2, 3\}^{\mathbb{N}}$, and certain transition matrices A, there is no multifractal rigidity, even in some generic sense. Namely, we show that it is impossible to completely characterize the equivalence class of φ from each entropy spectrum, except those spectra in a well-identified family. See Sect. 5.

We also describe appropriate procedures that allow us to obtain explicitly the equivalence classes determining a given spectrum.

The content of the paper is as follows. Section 2 contains several basic notions, including a discussion of the equivalence relation in (1). In Sect. 3 we introduce the class of locally constant functions, and describe some of its properties. Section 4 is dedicated to the study of multifractal rigidity, particular for locally constant functions on cylinders of length two on Markov chains with *two* symbols. In Sect. 5 we discuss the phenomenon of multifractal "nonrigidity" for locally constant functions on Markov chains with *three* symbols.

2 Entropy Spectrum of Gibbs Measures

2.1 Entropy Spectrum

Let $f: X \to X$ be a continuous map on a compact metric space. Let also μ be an f-invariant probability measure on X. Given a finite measurable partition ξ of X, we define the *local entropy* of μ at the point $x \in X$ by

$$h_\mu(f, \xi, x) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi_n(x)),$$

whenever the limit exists, where $\xi_n(x)$ is the element of the partition $\xi_n = \bigvee_{k=0}^{n-1} f^{-k}\xi$ containing x (mod 0). We define the *entropy spectrum (for the local entropies) of* μ by

$$\mathcal{E}(\alpha) = h(f | \{x \in X : h_\mu(f, \xi, x) = \alpha\}).$$

Let A be a $p \times p$ matrix whose entries a_{ij} are either 0 or 1, and let

$$\Sigma_A^+ = \{(i_1 i_2 \cdots) \in \{1, \dots, p\}^{\mathbb{N}} : a_{i_k i_{k+1}} = 1 \text{ for every } k \geq 1\}.$$

The associated shift map $\sigma: \Sigma_A^+ \to \Sigma_A^+$ is called a topological Markov chain. We always assume that there is a positive integer k such that all entries of A^k are positive (this happens if and only if $\sigma | \Sigma_A^+$ is topologically mixing). We also consider the distance in Σ_A^+ given by

$$d(i_1 i_2 \cdots, j_1 j_2 \cdots) = \sum_{k=1}^{+\infty} 2^{-k} |i_k - j_k|. \tag{2}$$

Given a continuous function $\varphi: \Sigma_A^+ \to \mathbb{R}$, a measure μ on Σ_A^+ is called a *Gibbs measure* for φ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \leq \frac{\mu(C_{i_1 \cdots i_n})}{\exp(-nP(\varphi) + \sum_{k=0}^{n-1} \varphi(\sigma^k w))} \leq D_2,$$

for every $w = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$, where

$$C_{i_1 \cdots i_n} = \{(j_1 j_2 \cdots) \in \Sigma_A^+ : j_k = i_k \text{ for } k = 1, \dots, n\} \tag{3}$$

(these sets are called cylinders), and $P(\varphi)$ is the topological pressure of φ with respect to σ. We recall that each Hölder continuous function φ has a unique σ-invariant probability Gibbs measure $\mu = \mu_\varphi$, which coincides with the unique equilibrium measure for φ.

We have the following description of the entropy spectrum of a Gibbs measure. Consider the function $T:\mathbb{R}\to\mathbb{R}$ defined by

$$T(q)=P(q\varphi)-qP(\varphi). \tag{4}$$

Proposition 1 (see [4]) *Let $\mu=\mu_\varphi$ be the equilibrium measure of a Hölder continuous function φ on Σ_A^+.*

1. *The function T is real analytic and satisfies $T'(q)\le 0$ and $T''(q)\ge 0$ for every $q\in\mathbb{R}$. Moreover, $T(0)=h(\sigma|\Sigma_A^+)$ and $T(1)=0$.*
2. *The range of the function $\alpha=-T'$ is the interval $[\alpha_1,\alpha_2]$, where $\alpha_1=\alpha(+\infty)$ and $\alpha_2=\alpha(-\infty)$. It coincides with the domain of the function $\mathcal{E}$.*
3. *We have $\mathcal{E}(\alpha(q))=T(q)+q\alpha(q)$ for every $q\in\mathbb{R}$.*
4. *If μ is not the measure of maximal entropy, then T and $\mathcal{E}$ are real analytic and strictly convex.*

It follows from properties 3 and 4 that the functions T and $\mathcal{E}$ form a Legendre pair. In particular, T can be recovered from $\mathcal{E}$ and vice-versa, i.e.,

$$\mathcal{E}(\alpha)=\inf_{q\in\mathbb{R}}(T(q)+q\alpha) \tag{5}$$

and

$$T(q)=\sup_{\alpha\in\mathbb{R}^+}(\mathcal{E}(\alpha)-q\alpha). \tag{6}$$

2.2 Equivalence Classes of Potentials

Given a Hölder continuous function φ, we would like to recover as much information as possible about the function from the entropy spectrum of its Gibbs measure. This is one of the main problems in the theory of multifractal rigidity (see [1]). Clearly, we cannot expect recovering the function φ itself since for example φ and $\varphi+c$, for any constant $c\in\mathbb{R}$, have the same Gibbs measure. We consider instead certain equivalence classes of potentials and our problem becomes to characterize the equivalence class corresponding to a given multifractal spectrum.

Two functions $\varphi,\psi:\Sigma_A^+\to\mathbb{R}$ are said to be *cohomologous* if there exist a continuous function $u:\Sigma_A^+\to\mathbb{R}$ and a constant $c\in\mathbb{R}$ such that

$$\varphi-\psi=u\circ\sigma-u+c. \tag{7}$$

Cohomologous functions have the same Gibbs measures, and thus their entropy spectra are also the same.

We denote by $\mathrm{Aut}(\Sigma_A^+)$ the family of automorphisms of (Σ_A^+,σ), that is, the homeomorphisms $\tau:\Sigma_A^+\to\Sigma_A^+$ such that $\tau\circ\sigma=\sigma\circ\tau$. We show that if two functions are related by an automorphism, then the entropy spectra of their Gibbs measures coincide.

Proposition 2 *Let (Σ_A^+,σ) be a topologically mixing Markov chain, and let $\varphi_1,\varphi_2:\Sigma_A^+\to\mathbb{R}$ be continuous functions. If $\varphi_2=\varphi_1\circ\tau$ for some automorphism $\tau\in\mathrm{Aut}(\Sigma_A^+)$, then*

$$P(q\varphi_1)=P(q\varphi_2)\ \textit{for any}\ q\in\mathbb{R}, \tag{8}$$

and thus the entropy spectra of the Gibbs measures of φ_1 and φ_2 are equal.

Proof Given a σ-invariant probability measure ν on Σ_A^+ we define a new measure on Σ_A^+ by $\nu_\tau(C) = \nu(\tau(C))$ for each measurable set C. Clearly, ν_τ is also a σ-invariant probability measure. We can easily verify that $h_\nu(\sigma, \xi) = h_{\nu_\tau}(\sigma, \tau^{-1}\xi)$, and thus $h_\nu(\sigma) = h_{\nu_\tau}(\sigma)$. We also have

$$\int_{\Sigma_A^+} \varphi_2 \, d\nu_\tau = \int_{\Sigma_A^+} (\varphi_1 \circ \tau) \, d(\nu \circ \tau) = \int_{\Sigma_A^+} \varphi_1 \, d\nu,$$

and hence, for every $q \in \mathbb{R}$,

$$h_\nu(\sigma) + q \int_{\Sigma_A^+} \varphi_1 \, d\nu = h_{\nu_\tau}(\sigma) + q \int_{\Sigma_A^+} \varphi_2 \, d\nu_\tau.$$

By the variational principle for the topological pressure, we conclude that the identity (8) is satisfied for any $q \in \mathbb{R}$.

In order to show that the entropy spectra are equal we notice that

$$T_1(q) := P(q\varphi_1) - qP(\varphi_1) = P(q\varphi_2) - qP(\varphi_2) =: T_2(q).$$

By Proposition 1, the entropy spectra of the Gibbs measures of φ_1 and φ_2 are respectively determined by the functions T_1 and T_2 (see (5)). Thus the spectra are equal. □

Due to the above observations, we can only discuss the multifractal rigidity problem of recovering φ from its spectrum $\mathcal{E}$ up to certain equivalence classes. To be precise, we say that two functions $\varphi_1, \varphi_2 \colon \Sigma_A^+ \to \mathbb{R}$ are *equivalent* if φ_1 and $\varphi_2 \circ \tau$ are cohomologous for some automorphism $\tau \in \mathrm{Aut}(\Sigma_A^+)$. We emphasize that in each of the corresponding equivalence classes any two functions have the same entropy spectrum.

3 Locally Constant Functions

A function $\varphi \colon \Sigma_A^+ \to \mathbb{R}$ is said to be *locally constant* if there exists $k \in \mathbb{N}$ such that $\varphi | C_{i_1 \cdots i_k}$ is constant for every $i_1, \ldots, i_k \in \{1, \ldots, p\}$ (see (3) for the definition of $C_{i_1 \cdots i_k}$). We denote by $\mathrm{LC}(k)$ the set of locally constant functions for a particular k. We will say that an equivalence class of functions C (see Sect. 2.2 for the definition) is an *equivalence class of* $\mathrm{LC}(k)$, or simply an $\mathrm{LC}(k)$ equivalence class, if $\varphi \in \mathrm{LC}(k)$ for some $\varphi \in C$, i.e., if there is a representative of C in $\mathrm{LC}(k)$.

The space $\mathrm{LC}(2)$ plays an important role since, as is well known, given a function $\varphi \in \mathrm{LC}(k)$ on a topological Markov chain Σ_A^+, there is a topological Markov chain Σ_B^+ equivalent to Σ_A^+ such that the image of φ under the equivalence is in $\mathrm{LC}(2)$. This shows that it is sufficient to consider locally constant functions on cylinders of length 2.

Proposition 3 *If $\varphi \colon \Sigma_A^+ \to \mathbb{R}$ is a function in* $\mathrm{LC}(k)$, *then there exist a Markov chain Σ_B^+ and a homeomorphism $\pi \colon \Sigma_A^+ \to \Sigma_B^+$ such that*

$$\pi \circ \sigma_A = \sigma_B \circ \pi \quad \textit{and} \quad \varphi \circ \pi^{-1} \in \mathrm{LC}(2). \tag{9}$$

Proof Although the result should be considered well-known it is hard to find a proof in the literature. For completeness we provide a simple construction. The idea is to group the elements of each sequence $(\alpha_0\alpha_1 \cdots) \in \Sigma_A^+$ in blocks of length $k-1$, and to obtain in this manner a new sequence that is in a new topological Markov chain Σ_B^+. We fix a bijection γ

between $\{1,\ldots,p\}^{k-1}$ and $\{1,\ldots,p^{k-1}\}$. For the $p\times p$ matrix $A=(a_{ij})$, the $p^{k-1}\times p^{k-1}$ matrix $B=(b_{ij})$ has entries

$$b_{ij}=\begin{cases}1, & \text{if } a_{\gamma^{-1}(i)_l\gamma^{-1}(i)_{l+1}}=1 \text{ for } l=1,\ldots,k-2,\\ & \text{and } a_{\gamma^{-1}(j)_l\gamma^{-1}(j)_{l+1}}=1 \text{ for } l=1,\ldots,k-2,\\ & \text{and } \gamma^{-1}(i)_m=\gamma^{-1}(j)_{m-1} \text{ for } m=2,\ldots,k-1,\\ 0, & \text{otherwise.}\end{cases}$$

We define the map π by

$$\pi(\alpha_0\alpha_1\cdots)=(\gamma(\alpha_0\cdots\alpha_{k-2})\gamma(\alpha_1\cdots\alpha_{k-1})\cdots).$$

One can easily show that $\pi(\Sigma_A^+)=\Sigma_B^+$: indeed, for $\alpha=(\alpha_1\alpha_2\cdots)\in\Sigma_A^+$ we have $b_{\pi(\alpha)_i\pi(\alpha)_{i+1}}=1$, since

$$\begin{aligned}&a_{\gamma^{-1}(\pi(\alpha)_i)_l\gamma^{-1}(\pi(\alpha)_i)_{l+1}}=a_{\alpha_{i+l-1}\alpha_{i+l}}=1 \text{ for } l=1,\ldots,k-2,\\ &a_{\gamma^{-1}(\pi(\alpha)_{i+1})_l\gamma^{-1}(\pi(\alpha)_{i+1})_{l+1}}=a_{\alpha_{i+l}\alpha_{i+l+1}}=1 \text{ for } l=1,\ldots,k-2,\\ &\gamma^{-1}(\pi(\alpha)_i)_m=\alpha_{i+m-1}=\gamma^{-1}(\pi(\alpha)_{i+1})_{m-1} \text{ for } m=2,\ldots,k-1.\end{aligned}$$

Furthermore, $\pi\colon\Sigma_A^+\to\Sigma_B^+$ is a homeomorphism since it maps cylinders of Σ_A^+ to cylinders of Σ_B^+, and the first identity in (9) can be easily checked. Since π maps cylinders of length k to cylinders of length 2, and $\varphi\in\mathrm{LC}(k)$ in Σ_A^+, we find that $\varphi\circ\pi^{-1}\in\mathrm{LC}(2)$ in Σ_B^+. $\square$

For functions in LC(2) there is an explicit expression for the topological pressure. To a function $\varphi\in\mathrm{LC}(2)$ we associate the $p\times p$ matrix

$$A(\varphi)=(a_{ij}\exp(\varphi|C_{ij}))_{i,j=1}^p. \tag{10}$$

We refer to $A(\varphi)$ as the matrix associated to φ, or simply the matrix of φ. The following is well know.

Proposition 4 *If $\varphi\in\mathrm{LC}(2)$, then $P(\varphi)=\log\rho_{A(\varphi)}$, where ρ_B is the spectral radius of the matrix B.*

We will consider functions $\varphi\in\mathrm{LC}(2)$ which are normalized, in the sense that $P(\varphi)=0$. By Proposition 4, these functions have a matrix $A(\varphi)$ with spectral radius equal to 1.

We now study the relations between the matrices $A(\varphi)$ and $A(\psi)$ of two equivalent LC(2) functions. We first introduce a family of automorphisms. For each permutation γ of $\{1,\ldots,p\}$ such that $a_{\gamma(i)\gamma(j)}=1$ whenever $a_{ij}=1$, we define an automorphism $\tau\colon\Sigma_A^+\to\Sigma_A^+$ by

$$\tau((\alpha_i)_{i\in\mathbb{N}})=(\gamma(\alpha_i))_{i\in\mathbb{N}}, \tag{11}$$

and we call it a *permutation automorphism.*

Theorem 1 *Let $\varphi,\psi\colon\Sigma_A^+\to\mathbb{R}$ be functions in* LC(2). *Then*:

1. *φ and ψ are cohomologous if and only if*

$$A(\varphi)=e^cD^{-1}A(\psi)D \tag{12}$$

for some positive diagonal matrix D and some constant $c\in\mathbb{R}$;

2. *$\varphi = \psi \circ \tau$ for some permutation automorphism τ if and only if*

$$A(\varphi) = P^{-1} A(\psi) P \tag{13}$$

for some permutation matrix P such that $A = P^{-1} A P$.

Proof We begin with an auxiliary result.

Lemma 1 *If $\varphi, \psi \in \mathrm{LC}(2)$ are cohomologous, then any continuous function u that satisfies (7) is in* $\mathrm{LC}(1)$.

Proof of the lemma We proceed by contradiction. If u were not in $\mathrm{LC}(1)$, then there would exist sequences $\alpha = (\alpha_1 \alpha_2 \cdots)$ and $\alpha' = (\alpha_1 \alpha_2' \cdots)$ with $\alpha_2 \neq \alpha_2'$ such that $u(\alpha) \neq u(\alpha')$. Take now β_0 with $a_{\beta_0 \alpha_1} = 1$, and consider the two sequences $\beta = (\beta_0 \alpha_1 \alpha_2 \cdots)$ and $\beta' = (\beta_0 \alpha_1 \alpha_2' \cdots)$ in $C_{\beta_0 \alpha_1}$. Since $\varphi, \psi \in \mathrm{LC}(2)$ we would have

$$\varphi(\beta) - \psi(\beta) = \varphi(\beta') - \psi(\beta'),$$

which is equivalent to

$$u(\alpha) - u(\alpha') = u(\beta) - u(\beta').$$

Similarly, for each $n \in \mathbb{N}$ we would obtain sequences $\beta^{(n)}, \beta'^{(n)} \in C_{\beta_{n-1} \cdots \beta_0 \alpha_1}$ such that

$$u(\beta^{(n)}) - u(\beta'^{(n)}) = u(\alpha) - u(\alpha') \neq 0.$$

Since $d(\beta^{(n)}, \beta'^{(n)}) \to 0$ when $n \to \infty$ (see (2)), we obtain a contradiction (note that u is continuous). This shows that $u \in \mathrm{LC}(1)$. □

We proceed with the proof of the theorem. Assume that $\varphi, \psi \in \mathrm{LC}(2)$ are cohomologous. By Lemma 1 there exist a continuous function $u \in \mathrm{LC}(1)$ and a constant $c \in \mathbb{R}$ satisfying (7). Thus,

$$\varphi | C_{ij} - \psi | C_{ij} = u | C_j - u | C_i + c,$$

and this implies that (12) holds with the diagonal matrix

$$D = \mathrm{diag}(\exp(u | C_i))_{i=1}^{p}. \tag{14}$$

In the other direction, we can easily verify that if (12) is satisfied for some positive diagonal matrix D and some constant c, then the functions φ and ψ are cohomologous (and satisfy (7) with u given by (14)).

We now assume that $\varphi = \psi \circ \tau$ for some permutation automorphism τ as in (11). Then $\varphi | C_{ij} = \psi | C_{\gamma(i)\gamma(j)}$, and the matrices $A(\varphi)$ and $A(\psi)$ are conjugated by the permutation matrix

$$P = (\delta_{i\gamma(j)})_{i,j=1}^{p}, \tag{15}$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, and $\delta_{\alpha\beta} = 0$ otherwise. We show that $A = P^{-1} A P$. Set $B = P^{-1} A P$. Since P^{-1} coincides with the transpose of P, we have

$$b_{ij} = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} (P^{-1})_{i\alpha} a_{\alpha\beta} P_{\beta j} = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \delta_{\alpha\gamma(i)} a_{\alpha\beta} \delta_{\beta\gamma(j)} = a_{\gamma(i)\gamma(j)},$$

where a_{ij} and b_{ij} are respectively the entries of A and B. If $a_{ij}=1$, then there exists $w=(ij\cdots)\in\Sigma_A^+$. The sequence $\tau(w)=(\gamma(i)\gamma(j)\cdots)$ is in Σ_A^+, and thus $a_{\gamma(i)\gamma(j)}=1$. On the other hand, if $a_{\gamma(i)\gamma(j)}=1$, then there exists $w'=(\gamma(i)\gamma(j)\cdots)\in\Sigma_A^+$. Since τ is a bijection, there also exists $w\in\Sigma_A^+$ with $\tau(w)=w'$. By the structure of the automorphism, we have $w=(ij\cdots)$, and thus $a_{ij}=1$. Therefore $a_{\gamma(i)\gamma(j)}=1$ if and only if $a_{ij}=1$. This shows that $b_{ij}=a_{ij}$ for each i and j, and hence $B=A$.

In the other direction, we can easily verify that if (13) is satisfied for some permutation matrix P such that $A=P^{-1}AP$, then $\varphi=\psi\circ\tau$ for some permutation automorphism τ (that can be read from the entries of P as in (15)). $\square$

We note that when the functions φ and ψ are cohomologous, the constant c in (12) is equal to $P(\varphi)-P(\psi)$. Thus, when the functions are normalized, i.e., $P(\varphi)=P(\psi)=0$, we have $c=0$ and $A(\varphi)=D^{-1}A(\psi)D$.

4 Multifractal Rigidity for Locally Constant Functions

4.1 LC(1) Functions on Full Markov Chains

We show that for full topological Markov chains Σ_n^+, the entropy spectrum $\mathcal{E}$ of an LC(1) equivalence class allows us to recover this class completely. This is the strongest possible multifractal rigidity.

Theorem 2 *Let (Σ_n^+,σ) be the full topological Markov chain with n symbols and $\mathcal{E}$ the entropy spectrum of an* LC(1) *equivalence class of functions. Then $\mathcal{E}$ completely characterizes the equivalence class.*

Proof Let $\varphi\in\mathrm{LC}(1)$ be an element of an LC(1) equivalence class, and write

$$\varphi|C_i=\log\alpha_i \text{ for } i=1,\dots,n. \tag{16}$$

Without loss of generality we assume that $\alpha_i\ge\alpha_{i+1}$ for $i=1,\dots,n-1$, and that $P(\varphi)=0$.

By (5) and (6) we can recover $\mathcal{E}$ from the function T in (4) and vice-versa. We show how to determine the equivalence class of φ from T, which is thus equivalent to determine the equivalence class from $\mathcal{E}$. We have

$$P(q\varphi)=\lim_{m\to\infty}\frac{1}{m}\log\sum_{i_1\cdots i_m}e^{q\sup_{C_{i_1\cdots i_m}}S_m\varphi(x)},$$

where $S_m\varphi(x)=\sum_{k=0}^{m-1}\varphi(\sigma^kx)$. Since

$$\sup_{C_{i_1\cdots i_m}}S_m\varphi(x)=\log(\alpha_{i_1}\cdots\alpha_{i_m}),$$

we obtain

$$\begin{aligned}P(q\varphi)&=\lim_{m\to\infty}\frac{1}{m}\log\sum_{i_1\cdots i_m}(\alpha_{i_1}\cdots\alpha_{i_m})^q\\&=\lim_{m\to\infty}\frac{1}{m}\log(\alpha_1^q+\cdots+\alpha_n^q)^m=\log(\alpha_1^q+\cdots+\alpha_n^q).\end{aligned}$$

Therefore

$$T(q) = \log(\alpha_1^q + \cdots + \alpha_n^q).$$

In order to determine the constants $\alpha_1, \ldots, \alpha_n$ from the function T, we use the following result.

Lemma 2 (see for example [7]) *Assume that the polynomial*

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{17}$$

has roots α_j, $j = 1, \ldots, n$, and set $\beta_k = \sum_{j=1}^n \alpha_j^k$ for $k \in \mathbb{N}$. Then

$$\beta_k + a_{n-1}\beta_{k-1} + \cdots + a_0\beta_{k-n} = 0, \quad \text{for } k > n,$$
$$\beta_k + a_{n-1}\beta_{k-1} + \cdots + a_{n-k+1}\beta_1 = -ka_{n-k}, \quad \text{for } 1 \le k \le n.$$

Set $\beta_k = \alpha_1^k + \cdots + \alpha_n^k = \exp T(k)$ for $k = 1, \ldots, n$, with $\alpha_1, \ldots, \alpha_n$ as in (16). By Lemma 2, we have

$$\begin{aligned}
-a_{n-1} &= \beta_1, \\
-2a_{n-2} &= \beta_2 + a_{n-1}\beta_1, \\
-3a_{n-3} &= \beta_3 + a_{n-2}\beta_2 + a_{n-3}\beta_1, \\
&\cdots \\
-na_0 &= \beta_n + a_{n-1}\beta_{n-1} + \cdots + a_1\beta_1.
\end{aligned} \tag{18}$$

These relations allow us to determine the coefficients $a_0, \ldots, a_{n-1}$ of the polynomial $p(x)$ in (17). To determine φ and hence its equivalence class we only need to compute the roots $\alpha_1, \ldots, \alpha_n$ of the polynomial. □

We note that the proof of Theorem 2 provides an algorithm to determine the LC(1) equivalence class of functions with a given spectrum on a full topological Markov chain (Σ_n^+, σ):

1. compute $\beta_k = \exp T(k)$ for $k = 1, \ldots, n$, and use the identities in (18) to compute $a_0, \ldots, a_{n-1}$;
2. compute the roots $\alpha_1, \ldots, \alpha_n$ of the polynomial $p(x)$ in (17);
3. the required LC(1) equivalence class contains the function $\varphi \in \text{LC}(1)$ satisfying (16).

4.2 LC(2) Functions on Markov Chains with 2 Symbols

We show in this section that for LC(2) functions on topological Markov chains with 2 symbols there is also a strong multifractal rigidity. However, this rigidity is not as strong as the one observed for LC(1) functions in Theorem 2.

Note that there is only three topological Markov chains with 2 symbols that are topologically mixing. One is the full Markov chain Σ_2^+. The other two have transition matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{19}$$

(and these two are equivalent by a permutation of symbols). In the case of Σ_2^+ we show that it is possible to characterize completely the LC(2) equivalence class from the entropy spectrum, except in a particular and well-identified case in which there are three possible equivalence classes (see Theorem 3). We note that our result is optimal since, as we shall see, the three classes have the same entropy spectra.

The case of Markov chains Σ_A^+ with a transition matrix A in (19) is considered in Theorem 4. In that case we will show that the entropy spectrum of an LC(2) equivalence class completely characterizes the equivalence class.

We start with the full topological Markov chain Σ_2^+.

Theorem 3 *Let* (Σ_2^+, σ) *be the full topological Markov chain with two symbols and let* $\mathcal{E}$ *be the entropy spectrum of an* LC(2) *equivalence class of functions. Then the equivalence class can be completely characterized from* $\mathcal{E}$, *except when*

$$T(q) = \log(\alpha^q + (1-\alpha)^q) \tag{20}$$

for some $\alpha \in (0, 1/2)$, *in which case there exist three distinct equivalence classes represented by the functions with matrices* (*see* (10))

$$\begin{pmatrix} 1-\alpha & 1-\alpha \\ \alpha & \alpha \end{pmatrix}, \qquad \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix}, \qquad \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}. \tag{21}$$

Proof We first notice that there always exists $\varphi \in \mathrm{LC}(2)$ in the equivalence class defining the spectrum $\mathcal{E}$ such that $P(\varphi) = 0$, and

$$\varphi|C_{11} = \log \alpha_{11}, \qquad \varphi|C_{12} = \log \alpha_{12}, \qquad \varphi|C_{21} = 0, \qquad \varphi|C_{22} = \log \alpha_{22},$$

with $\alpha_{11} \geq \alpha_{22}$. Indeed, consider a function φ in LC(2) with $P(\varphi) = 0$. Take now $u \in \mathrm{LC}(1)$ with $u|C_1 = -\varphi|C_{21}$ and $u|C_2 = 0$, and let

$$\psi = \varphi + u \circ \sigma - u.$$

Clearly, $\psi \in \mathrm{LC}(2)$, $P(\psi) = 0$, and

$$\psi|C_{21} = \varphi|C_{21} + u|C_1 - u|C_2 = 0.$$

We have (see (10))

$$A(\varphi) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 1 & \alpha_{22} \end{pmatrix},$$

and denoting by $\rho(q)$ the spectral radius of the matrix

$$A(q\varphi) = \begin{pmatrix} \alpha_{11}^q & \alpha_{12}^q \\ 1 & \alpha_{22}^q \end{pmatrix}, \tag{22}$$

it follows from Proposition 4 that $T(q) = \log \rho(q)$. We will consider the parameters

$$\overline{c} = \lim_{q \to +\infty} T'(q), \quad \underline{c} = \lim_{q \to -\infty} T'(q), \quad \text{and} \quad T'(0). \tag{23}$$

We first use $\overline{c}$ and $\underline{c}$ to determine functions that may originate the spectrum, and then we use $T'(0)$ to select the ones that actually can occur.

We start with an auxiliary result. We denote by M_σ the family of σ-invariant probability measures on Σ_2^+.

Lemma 3 *Let $\psi \in \mathrm{LC}(2)$ be a function with $\psi|C_{ij} = \beta_{ij}$ for $i, j \in \{1, 2\}$. Then*

$$\lim_{q\to+\infty} \int_{\Sigma_2^+} \psi \, d\mu_q = \max_{\nu\in M_\sigma} \int_{\Sigma_2^+} \psi \, d\nu = \max\{\beta_{11}, \tfrac{1}{2}(\beta_{12} + \beta_{21}), \beta_{22}\}, \tag{24}$$

where μ_q is the equilibrium measure of $q\psi$.

Proof of the lemma. We first show that

$$\lim_{q\to+\infty} \int_{\Sigma_2^+} \psi \, d\mu_q = \sup_{\nu\in M_\sigma} \int_{\Sigma_2^+} \psi \, d\nu. \tag{25}$$

Let $\nu \in M_\sigma$. By the variational principle for the topological pressure,

$$h_{\mu_q}(\sigma) + \int_{\Sigma_2^+} q\psi \, d\mu_q \geq h_\nu(\sigma) + \int_{\Sigma_2^+} q\psi \, d\nu.$$

Since $h_{\mu_q}(\sigma), h_\nu(\sigma) \leq \log 2$, dividing by q and letting $q \to +\infty$ we obtain

$$\lim_{q\to+\infty} \int_{\Sigma_2^+} \psi \, d\mu_q \geq \int_{\Sigma_2^+} \psi \, d\nu.$$

Since $\mu_q \in M_\sigma$ for any $q \in \mathbb{R}$ we thus obtain (25).

We now show that the supremum in (25) is in fact a maximum and that it is equal to the value in the right-hand side of (24). Let $\nu \in M_\sigma$. We have $\nu(C_1) = \nu(C_{11}) + \nu(C_{12})$ as well as

$$\nu(C_1) = \nu(\sigma^{-1}C_1) = \nu(C_{11}) + \nu(C_{21}).$$

Therefore, $\nu(C_{12}) = \nu(C_{21})$, and

$$\int_{\Sigma_2^+} \psi \, d\nu = \nu(C_{11})\beta_{11} + \nu(C_{12})(\beta_{12} + \beta_{21}) + \nu(C_{22})\beta_{22}.$$

Let us consider the function $\rho\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\rho(x, y, z) = x\beta_{11} + y(\beta_{12} + \beta_{21}) + z\beta_{22},$$

and the compact set

$$B = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 1 \text{ and } x, y, z \geq 0\}.$$

Clearly,

$$\sup_{\nu\in M_\sigma} \int_{\Sigma_2^+} \psi \, d\nu = \max\{\rho(x, y, z) : (x, y, z) \in B\}. \tag{26}$$

Furthermore, the maximum in (26) is attained at one of the vertices of B, namely $(1, 0, 0)$, $(0, 1/2, 0)$ or $(0, 0, 1)$. Thus,

$$\max\{\rho(x, y, z) : (x, y, z) \in B\} = \max\{\beta_{11}, \tfrac{1}{2}(\beta_{12} + \beta_{21}), \beta_{22}\}.$$

Let now $\delta(i_1\cdots i_k)\in M_\sigma$ be the measure supported on the periodic point $(i_1\cdots i_k i_1\cdots i_k\cdots)$. Setting

$$\nu_1=\delta(1),\quad \nu_2=\tfrac{1}{2}(\delta(12)+\delta(21)),\quad \nu_3=\delta(2)$$

we have

$$\int_{\Sigma_2^+}\psi\,d\nu_1=\beta_{11},\quad \int_{\Sigma_2^+}\psi\,d\nu_2=\frac{1}{2}(\beta_{12}+\beta_{21}),\quad \int_{\Sigma_2^+}\psi\,d\nu_3=\beta_2.$$

This shows that the supremum in (26) is in fact a maximum. This concludes the proof of the lemma. ☐

We recall that

$$T'(q)=\int_{\Sigma_2^+}\varphi\,d\mu_{q\varphi}$$

(see for example [6]). Hence, by Lemma 3,

$$\overline{c}=\max\left\{\log\alpha_{11},\frac{1}{2}\log\alpha_{12}\right\}\quad\text{and}\quad \underline{c}=\min\left\{\frac{1}{2}\log\alpha_{12},\log\alpha_{22}\right\}.$$

Since μ_0 is the Bernoulli measure $(1/2,1/2)$, we have

$$T'(0)=\frac{1}{4}(\log\alpha_{11}+\log\alpha_{12}+\log\alpha_{22}).\tag{27}$$

We now consider three cases:

1.
$$\log\alpha_{22}\le\frac{1}{2}\log\alpha_{12}\le\log\alpha_{11},\tag{28}$$
in which case $\overline{c}=\log\alpha_{11}$ and $\underline{c}=\log\alpha_{22}$;
2.
$$\frac{1}{2}\log\alpha_{12}\le\log\alpha_{22}\le\log\alpha_{11},\tag{29}$$
in which case $\overline{c}=\log\alpha_{11}$ and $2\underline{c}=\log\alpha_{12}$;
3.
$$\log\alpha_{22}\le\log\alpha_{11}\le\frac{1}{2}\log\alpha_{12},\tag{30}$$
in which case $2\overline{c}=\log\alpha_{12}$ and $\underline{c}=\log\alpha_{22}$.

In the three cases two of the numbers α_{11}, α_{12} and α_{22} are determined by $\overline{c}$ and $\underline{c}$, although we are not able to say which ones. In the first case α_{11} and α_{22} are determined, in the second case α_{11} and α_{12}, and in the third case α_{12} and α_{22}. In order to determine the third number (among α_{11}, α_{12} and α_{22}) we note that $A(\varphi)$ is a positive matrix, and thus it has a positive eigenvalue greater in absolute value than the second one. Since the spectral radius of $A(\varphi)$ is $\exp P(\varphi)=1$ (see Proposition 4), the maximal eigenvalue of the matrix is thus exactly 1.

This will be used to determine the third number. Since $\rho_{A(\varphi)} = 1$, we can easily verify that the matrix $A(\varphi)$ must be, in each case,

$$\begin{array}{ccc} \textit{Case 1} & \textit{Case 2} & \textit{Case 3} \\ \begin{pmatrix} e^{\overline{c}} & (1-e^{\overline{c}})(1-e^{\underline{c}}) \\ 1 & e^{\underline{c}} \end{pmatrix} & \begin{pmatrix} e^{\overline{c}} & e^{2\underline{c}} \\ 1 & 1-\frac{e^{2\underline{c}}}{1-e^{\overline{c}}} \end{pmatrix} & \begin{pmatrix} 1-\frac{e^{2\overline{c}}}{1-e^{\underline{c}}} & e^{2\overline{c}} \\ 1 & e^{\underline{c}} \end{pmatrix} \end{array}. \tag{31}$$

We consider the three cases and determine which values $\overline{c}$ and $\underline{c}$ may attain:

1. In this case $e^{\underline{c}} < 1$. Indeed, if we would have $e^{\underline{c}} > 1$ then the trace of $A(\varphi)$ would be greater than 2, and thus its spectral radius would be greater than 1. Furthermore, if it would be $e^{\underline{c}} = 1$ then we would have $\alpha_{12} = 0$, which is impossible. We also have $e^{\overline{c}} < 1$ since otherwise $\alpha_{12} \leq 0$ which is again impossible. It follows from (28) that

$$e^{2\underline{c}} \leq (1-e^{\overline{c}})(1-e^{\underline{c}}) \leq e^{2\overline{c}},$$

which is equivalent to

$$\frac{e^{\underline{c}} - 1 + \sqrt{(1-e^{\underline{c}})^2 + 4(1-e^{\underline{c}})}}{2} \leq e^{\overline{c}} \leq 1 - \frac{e^{2\underline{c}}}{1-e^{\underline{c}}}. \tag{32}$$

2. In this case $e^{\overline{c}} < 1$. Indeed, similarly, if we would have $e^{\overline{c}} > 1$ then the trace of $A(\varphi)$ would be greater than 2, and thus its spectral radius would be greater than 1. Furthermore, if it would be $e^{\overline{c}} = 1$ then α_{22} would be undefined. Since $\underline{c} \leq \overline{c}$ we also have $e^{\underline{c}} < 1$. It follows from (29) that

$$e^{\underline{c}} \leq 1 - \frac{e^{2\underline{c}}}{1-e^{\overline{c}}} \leq e^{\overline{c}},$$

which is equivalent to

$$1 - e^{\underline{c}} \leq e^{\overline{c}} \leq 1 - \frac{e^{2\underline{c}}}{1-e^{\underline{c}}}. \tag{33}$$

3. Proceeding as in Case 2 we find that $e^{\underline{c}} < 1$. Furthermore, we must have $e^{\overline{c}} < 1$ since otherwise α_{11} would be negative, which is impossible. It follows from (30) that

$$e^{\underline{c}} \leq 1 - \frac{e^{2\overline{c}}}{1-e^{\underline{c}}} \leq e^{\overline{c}},$$

which is equivalent to

$$\frac{e^{\underline{c}} - 1 + \sqrt{(1-e^{\underline{c}})^2 + 4(1-e^{\underline{c}})}}{2} \leq e^{\overline{c}} \leq 1 - e^{\underline{c}}. \tag{34}$$

It can be easily verified that for $e^{\underline{c}} > 1/2$ the functions

$$\frac{e^{\underline{c}} - 1 + \sqrt{(1-e^{\underline{c}})^2 + 4(1-e^{\underline{c}})}}{2}, \quad 1 - \frac{e^{2\underline{c}}}{1-e^{\underline{c}}} \quad \text{and} \quad 1 - e^{\underline{c}}$$

are smaller than $e^{\underline{c}}$. Since $e^{\overline{c}}$ is smaller than or equal to some of these functions, we conclude that for $e^{\underline{c}} > 1/2$ we would have $e^{\overline{c}} < e^{\underline{c}}$, which is impossible. Thus we must have $e^{\underline{c}} \leq 1/2$.

Note that for $e^{\underline{c}} \in [0, 1/2]$,

$$\frac{e^{\underline{c}} - 1 + \sqrt{(1-e^{\underline{c}})^2 + 4(1-e^{\underline{c}})}}{2} \le 1 - e^{\underline{c}} \le 1 - \frac{e^{2\underline{c}}}{1-e^{\underline{c}}},$$

with equalities when $e^{\underline{c}} = 1/2$.

The three inequalities in (32–34) can be simultaneously satisfied only when

$$e^{\overline{c}} = 1 - e^{\underline{c}}. \tag{35}$$

In this case we are in a situation in which it is impossible to completely recover the equivalence class. Instead, we have three classes represented by the functions φ with matrices

$$\begin{array}{ccc} \textit{Case 1} & \textit{Case 2} & \textit{Case 3} \\ \begin{pmatrix} 1-e^{\underline{c}} & e^{\underline{c}}(1-e^{\underline{c}}) \\ 1 & e^{\underline{c}} \end{pmatrix} & \begin{pmatrix} 1-e^{\underline{c}} & e^{2\underline{c}} \\ 1 & 1-e^{\underline{c}} \end{pmatrix} & \begin{pmatrix} e^{\underline{c}} & (1-e^{\underline{c}})^2 \\ 1 & e^{\underline{c}} \end{pmatrix}. \end{array} \tag{36}$$

When $e^{\underline{c}} = 1/2$ the three matrices coincide and thus we recover a single class. When $e^{\underline{c}} \neq 1/2$ the functions are cohomologous to the ones in (21) with $\alpha = e^{\underline{c}}$. We now show that the three functions are not equivalent although they have the same entropy spectrum.

Lemma 4 *Let* (Σ_2^+, σ) *be the full topological Markov chain with two symbols and let* $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{LC}(2)$ *be the functions with matrices respectively as in* (21) *for some* $\alpha \in (0, 1/2)$. *Then the equilibrium measures of* φ_1, φ_2, *and* φ_3 *have the same entropy spectrum but the functions are not equivalent.*

Proof of the lemma Computing the spectral radius of the matrices $A(q\varphi_i)$ we obtain

$$P(q\varphi_i) = \log(\alpha^q + (1-\alpha)^q), \quad i = 1, 2, 3.$$

Hence, by (5), the entropy spectra of the three functions coincide.

We now show that the functions are not equivalent. Note that any $\tau \in \mathrm{Aut}(\Sigma_2^+)$ transforms fixed points of σ^n into fixed points of σ^n. In particular, setting $\gamma_1 = (11\cdots)$ and $\gamma_2 = (22\cdots)$, we obtain $\{\tau(\gamma_1), \tau(\gamma_2)\} = \{\gamma_1, \gamma_2\}$. Thus, if two functions $\psi_1, \psi_2 \in \mathrm{LC}(2)$ with $P(\psi_1) = P(\psi_2)$ are equivalent, then

$$\{\psi_1(\gamma_1), \psi_1(\gamma_2)\} = \{\psi_2(\gamma_1), \psi_2(\gamma_2)\}.$$

Since the functions φ_i, $i = 1, 2, 3$ have the same topological pressure and the sets $\{\varphi_i(\gamma_1), \varphi_i(\gamma_2)\}$, $i = 1, 2, 3$ are distinct, we conclude that φ_1, φ_2, and φ_3 cannot be equivalent. □

We now assume that (35) does not occur, i.e., that $e^{\overline{c}} \neq 1 - e^{\underline{c}}$. There are two possibilities: when

$$1 - e^{\underline{c}} < e^{\overline{c}} \le 1 - \frac{e^{2\underline{c}}}{1-e^{\underline{c}}}, \tag{37}$$

we are in Case 1 or Case 2 (see (28) and (29)), and when

$$\frac{e^{\underline{c}} - 1 + \sqrt{(1-e^{\underline{c}})^2 + 4(1-e^{\underline{c}})}}{2} \le e^{\overline{c}} < 1 - e^{\underline{c}}, \tag{38}$$

we are in Case 1 or Case 3 (see (28) and (30)). Using the parameter $T'(0)$ we will determine in which case we are, and thus we will be able to identify a unique equivalence class.

Assume first that (37) holds. In Case 1 (see (27)) we have

$$e^{4T'(0)} = e^{\overline{c}} e^{\underline{c}} (1 - e^{\overline{c}})(1 - e^{\underline{c}}), \tag{39}$$

and in Case 2 we have

$$e^{4T'(0)} = e^{\overline{c}} e^{2\underline{c}} \left(1 - \frac{e^{2\underline{c}}}{1 - e^{\overline{c}}}\right). \tag{40}$$

If only one of these identities holds, then we identify exactly the case in which we are and thus the equivalence class. Otherwise, when both identities in (39) and (40) are satisfied we must have

$$e^{\overline{c}} = 1 - e^{\underline{c}} \quad \text{or} \quad e^{\overline{c}} = 1 - \frac{e^{2\underline{c}}}{1 - e^{\underline{c}}}, \tag{41}$$

which are conditions at the boundary of the region of Case 2. The first identity in (41) was already analyzed and corresponds to the situation when we obtain three equivalence classes. When the second identity in (41) holds, the equivalence classes in Case 1 and Case 2 are equal, and contain the function with matrix

$$\begin{pmatrix} 1 - \frac{e^{2\underline{c}}}{1 - e^{\underline{c}}} & e^{2\underline{c}} \\ 1 & e^{\underline{c}} \end{pmatrix}.$$

Thus, when (37) holds, since $e^{\overline{c}} \neq 1 - e^{\underline{c}}$ we identify a unique LC(2) equivalence class that generates the spectrum.

The situation when (38) holds is similar. Namely, in Case 1 we have

$$e^{4T'(0)} = e^{\overline{c}} e^{\underline{c}} (1 - e^{\overline{c}})(1 - e^{\underline{c}}), \tag{42}$$

and in Case 3 we have

$$e^{4T'(0)} = e^{2\overline{c}} e^{\underline{c}} \left(1 - \frac{e^{2\overline{c}}}{1 - e^{\underline{c}}}\right). \tag{43}$$

If only one of these identities holds, again we identify exactly one equivalence class that generates the spectrum. Otherwise, when both identities in (42) and (43) are satisfied we must have

$$e^{\overline{c}} = 1 - e^{\underline{c}} \quad \text{or} \quad e^{\underline{c}} = 1 - \frac{e^{2\overline{c}}}{1 - e^{\overline{c}}}, \tag{44}$$

now corresponding to the boundary of the region of Case 3. When the second identity in (44) holds, the equivalence classes in Case 1 and Case 3 are equal, and contain the function with matrix

$$\begin{pmatrix} e^{\overline{c}} & e^{2\overline{c}} \\ 1 & 1 - \frac{e^{2\underline{c}}}{1 - e^{\underline{c}}} \end{pmatrix}.$$

Thus, when (38) holds, since $e^{\overline{c}} \neq 1 - e^{\underline{c}}$ we identify a unique LC(2) equivalence class that generates the spectrum. This concludes the proof of the theorem. □

It follows from the proof of Theorem 3 that the case with three equivalence classes (see (20)) occurs if and only if

$$\exp \lim_{q\to+\infty} T'(q) + \exp \lim_{q\to-\infty} T'(q) = 1. \tag{45}$$

By Proposition 1, if $[\alpha_1, \alpha_2]$ is the domain of the entropy spectrum $\mathcal{E}$, the identity (45) can be written in the form $e^{-\alpha_1} + e^{-\alpha_2} = 1$.

The proof of Theorem 3 also provides an algorithm to determine the equivalence classes with a given spectrum:

1. compute $\overline{c}$, $\underline{c}$ and $T'(0)$;
2. if $e^{\overline{c}} + e^{\underline{c}} = 1$, then we obtain three equivalence classes, represented by the functions with matrices in (36);
3. if (39) holds, then the unique equivalence class is represented by the first matrix in (31);
4. if (40) holds, then the unique equivalence class is represented by the second matrix in (31);
5. if (43) holds, then the unique equivalence class is represented by the third matrix in (31).

We now consider the case of the topological Markov chain Σ_A^+ with two symbols, with transition matrix $A = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ or $A = \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$.

Theorem 4 *Let (Σ_A^+, σ) be the topological Markov chain with two symbols, with transition matrix $A = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ or $A = \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$, and let $\mathcal{E}$ be the entropy spectrum of an* LC(2) *equivalence class of functions. Then we can completely characterize the equivalence class from the spectrum $\mathcal{E}$.*

Proof We only consider the case when $A = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ since the other one is entirely similar. Let $\varphi \in \mathrm{LC}(2)$ be an element of the equivalence class defining the spectrum $\mathcal{E}$ such that $P(\varphi) = 0$, and

$$\varphi|C_{11} = \log\alpha_{11}, \qquad \varphi|C_{12} = \log\alpha_{12}, \qquad \varphi|C_{21} = 0$$

(as in the proof of Theorem 3 we can easily verify that there always exists a function φ with these properties). Recall that we can compute $T(q) = P(q\varphi)$ from the spectrum $\mathcal{E}$ (see (6)). We proceed in a similar manner to that in the proof of Theorem 3, considering now the two parameters $\overline{c}$ and $\underline{c}$ in (23). As in the proof of Lemma 3 we can show that

$$\overline{c} = \max\left\{\log\alpha_{11}, \frac{1}{2}\log\alpha_{12}\right\} \quad \text{and} \quad \underline{c} = \min\left\{\log\alpha_{11}, \frac{1}{2}\log\alpha_{12}\right\}.$$

We can easily verify that there are two possibilities for the matrix $A(\varphi)$, namely

$$\begin{array}{cc} \textit{Case 1} & \textit{Case 2} \\ \begin{pmatrix} e^{\overline{c}} & e^{2\underline{c}} \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} e^{\underline{c}} & e^{2\overline{c}} \\ 1 & 0 \end{pmatrix} \end{array}. \tag{46}$$

Since $P(\varphi) = 0$, the function φ is represented by a matrix with spectral radius 1, and thus the matrices in (46) must have 1 as an eigenvalue. Therefore, in Case 1 we have

$$e^{\overline{c}} + e^{2\underline{c}} = 1, \tag{47}$$

and in Case 2 we have

$$e^{2\overline{c}} + e^{\underline{c}} = 1. \tag{48}$$

In order that the two identities (47) and (48) are satisfied simultaneously the number $e^{\underline{c}}$ must be 0, $(-1+\sqrt{5})/2$ or 1. It cannot be 0, and also cannot be 1 since $e^{\overline{c}}$ would then be 0. Thus we must have $e^{\underline{c}} = (-1+\sqrt{5})/2$. We can easily verify that for this value of $e^{\underline{c}}$ the two matrices in (46) are equal.

To determine the equivalence class we only need to verify which of the identities in (47) and (48) is satisfied. If (47) holds, then the equivalence class is represented by the first matrix in (46). Otherwise, if (48) holds, then the equivalence class is represented by the second matrix in (46). When both identities hold, as we observed above, the two matrices in (46) are equal, and thus we obtain a unique equivalence class. Therefore, for each spectrum of an LC(2) equivalence class there is a unique class with that spectrum. □

The proof of Theorem 4 also provides an algorithm to obtain the LC(2) equivalence class from the spectrum. When $A = \left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$ the algorithm is as follows:

1. compute $\overline{c}$ and $\underline{c}$;
2. if $e^{\overline{c}} + e^{2\underline{c}} = 1$, then the equivalence class is represented by the first matrix in (46);
3. if $e^{2\overline{c}} + e^{\underline{c}} = 1$, then the equivalence class is represented by the second matrix in (46).

5 Multifractal Nonrigidity

We give in this section an explicit example of a topological Markov chain with *three* symbols for which there is no multifractal rigidity, even more in some generic sense.

Let $\varphi \in \mathrm{LC}(2)$ be a function with $P(\varphi) = 0$. Recall that by Proposition 4,

$$T(q) = P(q\varphi) = \log \rho_{A(q\varphi)}.$$

Given a square matrix $A = (a_{ij})$ we consider the characteristic polynomial

$$p_A(z,q) = \det(z\,\mathrm{Id} - A^{(q)}), \tag{49}$$

where $A^{(q)} = (a_{ij}^q)$. We note that if $p_{A(\varphi)}$ is known explicitly, then T can be computed explicitly. This follows from the identity

$$p_{A(\varphi)}(z,q) = \det(z\,\mathrm{Id} - A(\varphi)^{(q)}) = \det(z\,\mathrm{Id} - A(q\varphi))$$

(see (22)). Thus, if we know $p_{A(\varphi)}$ explicitly, then the entropy spectrum $\mathcal{E}$ can also be computed explicitly (see (5)). Nevertheless, we will see that in general the knowledge of $p_{A(\varphi)}$ is insufficient to determine the equivalence class of φ. More precisely, we will exhibit functions in distinct equivalence classes that have the same characteristic polynomial, and thus the same entropy spectrum. This shows that the knowledge of T (and thus the knowledge of $\mathcal{E}$) is insufficient to determine the equivalence class of φ.

Theorem 5 *Let (Σ_A^+, σ) be the topological Markov chain with transition matrix*

$$A = \begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0 \end{pmatrix}, \tag{50}$$

and let $\varphi, \psi \in \mathrm{LC}(2)$ *be functions satisfying* $P(\varphi) = P(\psi) = 0$ *with matrices*

$$A(\varphi) = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 1 & 0 & \alpha_{23} \\ 1 & \alpha_{32} & 0 \end{pmatrix} \quad \textit{and} \quad A(\psi) = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 1 & 0 & \frac{\alpha_{13}}{\alpha_{12}}\alpha_{32} \\ 1 & \frac{\alpha_{12}}{\alpha_{13}}\alpha_{23} & 0 \end{pmatrix}, \tag{51}$$

where $\alpha_{12} > \alpha_{13} > \alpha_{23}\alpha_{32}$ *and* $\alpha_{13}\alpha_{32} \neq \alpha_{12}\alpha_{23}$. *Then the functions* φ *and* ψ *have the same characteristic polynomial but are not equivalent.*

Proof We can easily verify that $p_{A(\varphi)} = p_{A(\psi)} = p$, where

$$p(z, q) = z^3 - (\alpha_{12}^q + \alpha_{13}^q + (\alpha_{23}\alpha_{32})^q)z + (\alpha_{12}\alpha_{23})^q + (\alpha_{13}\alpha_{32})^q.$$

In order to show that the functions φ and ψ are not equivalent we first observe that $\mathrm{Aut}(\Sigma_A^+) \approx S_3$, where S_3 is the permutation group of 3 elements (see Example 2.19 in [3]). Notice that to each permutation γ of $\{1, 2, 3\}$ corresponds a permutation automorphism in $\mathrm{Aut}(\Sigma_A^+)$ (see (11)). Therefore, the automorphisms of Σ_A^+ are precisely the permutation automorphisms.

We proceed by contradiction. Assume that φ and ψ are equivalent, i.e., that there exist $\tau \in \mathrm{Aut}(\Sigma_A^+)$ obtained from a permutation γ as in (11), and a continuous function $u\colon \Sigma_A^+ \to \mathbb{R}$ such that

$$\varphi \circ \tau = \psi + u \circ \sigma - u. \tag{52}$$

Since φ and ψ are in LC(2), the function $\varphi \circ \tau$ is also in LC(2), and u is in LC(1) (see Lemma 1). Thus we can write $u|C_i = \log d_i$ with $d_i > 0$ for $i = 1, 2, 3$. The identity (52) can be written in matrix form as

$$\begin{pmatrix} 0 & \alpha_{\gamma(1)\gamma(2)} & \alpha_{\gamma(1)\gamma(3)} \\ \alpha_{\gamma(2)\gamma(1)} & 0 & \alpha_{\gamma(2)\gamma(3)} \\ \alpha_{\gamma(3)\gamma(1)} & \alpha_{\gamma(3)\gamma(2)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d_1}{d_2}\alpha_{12} & \frac{d_1}{d_3}\alpha_{13} \\ \frac{d_2}{d_1} & 0 & \frac{d_2}{d_3}\frac{\alpha_{13}}{\alpha_{12}}\alpha_{32} \\ \frac{d_3}{d_1} & \frac{d_3}{d_2}\frac{\alpha_{12}}{\alpha_{13}}\alpha_{23} & 0 \end{pmatrix}. \tag{53}$$

We note that

$$\begin{aligned} \alpha_{\gamma(1)\gamma(2)}\alpha_{\gamma(2)\gamma(1)} &= \alpha_{12}, \\ \alpha_{\gamma(1)\gamma(3)}\alpha_{\gamma(3)\gamma(1)} &= \alpha_{13}, \\ \alpha_{\gamma(2)\gamma(3)}\alpha_{\gamma(3)\gamma(2)} &= \alpha_{23}\alpha_{32}. \end{aligned} \tag{54}$$

The matrix in the left-hand side of (53) can be written in the form $P^{-1}A(\varphi)P$ where P is the permutation matrix in (15). Since the set of entries of $PA(\varphi)P^{-1}$ is the same as the set of entries of $A(\varphi)$, we can easily verify that the identities in (54) together with the hypothesis $\alpha_{12} > \alpha_{13} > \alpha_{23}\alpha_{32}$ imply that P is the identity matrix (and thus τ is the identity automorphism).

Indeed, if $\gamma(1) = 2$ then $\alpha_{2\gamma(2)}\alpha_{\gamma(2)2} = \alpha_{12}$. Since $\alpha_{23}\alpha_{32} < \alpha_{12}$ it cannot be $\gamma(2) = 3$. Therefore, $\gamma(2) = 1$ and $\gamma(3) = 3$. But from the second identity in (54) we then should have $\alpha_{23}\alpha_{32} = \alpha_{13}$, which contradicts to the hypotheses in the theorem. We can show in a similar manner that $\gamma(1) = 3$ yields a contradiction. Indeed, if $\gamma(1) = 3$ then from the first identity in (54) we would have $\alpha_{3\gamma(2)}\alpha_{\gamma(2)3} = \alpha_{12}$. It follows from the hypotheses that it cannot be $\gamma(2) = 2$, and thus we must have $\gamma(2) = 1$ and $\gamma(3) = 2$. Then the second identity in (54) gives $\alpha_{32}\alpha_{23} = \alpha_{13}$ which contradicts again the hypotheses in the theorem. Therefore, we must have $\gamma(1) = 1$, and the first identity in (54) gives $\alpha_{1\gamma(2)}\alpha_{\gamma(2)1} = \alpha_{12}$. If $\gamma(2) = 3$ then we obtain $\alpha_{13} = \alpha_{12}$ which is forbidden by the hypotheses. Thus we must have $\gamma(2) = 2$ and $\gamma(3) = 3$. This shows that $\gamma = \mathrm{Id}$, and thus τ is the identity automorphism.

Equation (53) thus reduces to

$$\begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 1 & 0 & \alpha_{23} \\ 1 & \alpha_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d_1}{d_2}\alpha_{12} & \frac{d_1}{d_3}\alpha_{13} \\ \frac{d_2}{d_1} & 0 & \frac{d_2}{d_3}\frac{\alpha_{13}}{\alpha_{12}}\alpha_{32} \\ \frac{d_3}{d_1} & \frac{d_3}{d_2}\frac{\alpha_{12}}{\alpha_{13}}\alpha_{23} & 0 \end{pmatrix}.$$

This implies that $d_1 = d_2 = d_3$ and hence u is constant. Therefore, $\psi = \varphi$. On the other hand, since $\alpha_{13}\alpha_{32} \neq \alpha_{12}\alpha_{23}$ we must have $\psi \neq \varphi$. This contradiction shows that the functions ψ and φ cannot be equivalent, contrarily to what was assumed above (see (52)). □

We note that for the matrix A in (50) all entries of A^2 are positive, and thus the associated Markov chain is topologically mixing.

Theorem 5 tell us that given a characteristic polynomial of a generic LC(2) function, we can only recover information that allows us to identify at least two distinct equivalence classes. Since the function T possesses less information than the characteristic polynomial, it could happen that using solely the information given by T we could in general obtain more equivalence classes than the two above obtained from the characteristic polynomial. However, we now show that T contains sufficient information to obtain exactly the two equivalence classes.

Theorem 6 *Let (Σ_A^+, σ) be the topological Markov chain with transition matrix A as in* (50), *and let $\mathcal{E}$ be the entropy spectrum of an* LC(2) *function. Then there are two distinct* LC(2) *equivalence classes that generate the spectrum $\mathcal{E}$, except in a particular case in which there is a unique* LC(2) *equivalence class generating the spectrum.*

Proof We separate the proof into several steps. We start with some auxiliary results. Let (Σ_A^+, σ) be a topologically mixing Markov chain with p symbols, and let $\eta \in \mathrm{LC}(2)$ be a function with matrix $A(\eta) = (c_{ij})_{i,j=1}^p$. For $i = 1, \ldots, p$ we set

$$E_i = \{c_{j_1\tau(j_1)} \cdots c_{j_i\tau(j_i)} : 1 \leq j_1 < \cdots < j_i \leq p \text{ and } \tau \in P_i\},$$

where P_i is the family of permutations of $\{j_1, \ldots, j_i\}$. We also set

$$\overline{\lambda} = \max\{\alpha^{1/i} : i = 1, \ldots, p \text{ and } \alpha \in E_i\}, \tag{55}$$

$$\underline{\lambda} = \min\{\alpha^{1/i} : i = 1, \ldots, p \text{ and } \alpha \in E_i\}, \tag{56}$$

and let $\rho(q)$ be the spectral radius of the matrix

$$A(q\eta) = (c_{ij}^q)_{i,j=1}^p. \tag{57}$$

Lemma 5 *The limits*

$$\overline{\rho} = \lim_{q\to+\infty} \frac{\rho(q)}{\overline{\lambda}^q} \quad \text{and} \quad \underline{\rho} = \lim_{q\to-\infty} \frac{\rho(q)}{\underline{\lambda}^q} \tag{58}$$

exist and are nonzero. Moreover, if there is only one value $\alpha^{1/i}$ equal to $\overline{\lambda}$ (respectively, $\underline{\lambda}$), then $\overline{\rho} = 1$ (respectively, $\underline{\rho} = 1$).

Proof of the lemma The characteristic polynomial of $A(\eta)$ (see (49)) is given by

$$p(z,q) = z^p + \sum_{i=1}^{p} \sum_{\alpha\in E_i} (-1)^i (-1)^{\sigma(\alpha)} \alpha^q z^{p-i}, \tag{59}$$

where $\sigma(\alpha)$ is the sign of the permutation associated to α. Since $p(\rho(q),q)=0$, taking $z=\rho(q)$ and dividing in (59) by $\rho(q)^p$ we obtain

$$1+\sum_{i=1}^{p}(-1)^i \frac{\sum_{\alpha\in E_i}(-1)^{\sigma(\alpha)}\alpha^q}{\rho(q)^i} = 0. \tag{60}$$

For each $i=1,\dots,p$, set $\alpha_i = \max\{\alpha : \alpha\in E_i\}$. Let n_i be the number of times that the maximum is attained, and let $\alpha_{i,1},\dots,\alpha_{i,n_i}$ be the elements of E_i equal to α_i. Then

$$\lim_{q\to+\infty} \frac{\sum_{\alpha\in E_i}(-1)^{\sigma(\alpha)}\alpha^q}{\rho(q)^i} = \lim_{q\to+\infty} \sum_{k=1}^{n_i} (-1)^{\sigma(\alpha_{i,k})} \frac{\alpha_i^q}{\rho(q)^i} = \sum_{k=1}^{n_i}(-1)^{\sigma(\alpha_{i,k})} \lim_{q\to+\infty}\left[\frac{(\alpha_i^{1/i})^q}{\rho(q)}\right]^i.$$

Thus, letting $q\to+\infty$ in (60) yields

$$1+\sum_{i=1}^{p}\sum_{k=1}^{n_i}(-1)^i(-1)^{\sigma(\alpha_{i,k})} \lim_{q\to+\infty}\left[\frac{(\alpha_i^{1/i})^q}{\rho(q)}\right]^i = 0.$$

Since

$$\overline{\lambda} = \max\{\alpha_i^{1/i} : i=1,\dots,p\},$$

denoting by J the set of integers $j\in\{1,\dots,p\}$ such that $\alpha_j^{1/j}=\overline{\lambda}$ we obtain

$$1+\sum_{j\in J}\sum_{k=1}^{n_j}(-1)^j(-1)^{\sigma(\alpha_{j,k})} \lim_{q\to+\infty}\left[\frac{\overline{\lambda}^q}{\rho(q)}\right]^j = 0. \tag{61}$$

We conclude that the first limit in (58) exists and is nonzero.

If there is only one value $\alpha^{1/i}$ equal to $\overline{\lambda}$ with $\alpha\in E_i$, then J has a single element $j\in\{1,\dots,p\}$, $n_j=1$, and (61) reduces to

$$1+(-1)^j(-1)^{\sigma(\alpha_j)} \lim_{q\to+\infty}\left[\frac{\overline{\lambda}^q}{\rho(q)}\right]^j = 0.$$

This implies that the first limit in (58) has absolute value equal to 1. Since $\rho(q)$ and $\overline{\lambda}$ are positive the limit is 1.

The case when $q\to-\infty$ can be treated in a similar manner, by interchanging the roles of $\overline{\lambda}$ and $\underline{\lambda}$. □

The following result gives a formula for $T'(q)$.

Lemma 6 *If p is the characteristic polynomial of $A(\eta)$ (see (59)), then*

$$T'(q) = -\frac{\frac{\partial}{\partial q}p(\rho(q),q)}{\rho(q)\frac{\partial}{\partial z}p(\rho(q),q)} - P(\eta). \tag{62}$$

Proof of the lemma We have

$$T(q) = \log \rho(q) - qP(\eta),$$

and thus

$$T'(q) = \frac{\rho'(q)}{\rho(q)} - P(\eta). \tag{63}$$

Since $p(\rho(q), q) = 0$, differentiating with respect to q we obtain a formula for $\rho'(q)$ that substituted in (63) yields (62). □

We now compute the limit of $T'(q)$ as $q \to \pm\infty$.

Lemma 7 *We have*

$$\lim_{q\to+\infty} T'(q) = \log \overline{\lambda} - P(\eta) \quad \textit{and} \quad \lim_{q\to-\infty} T'(q) = \log \underline{\lambda} - P(\eta).$$

Proof of the lemma As in the proof of Lemma 5, let n_i be the number of elements $\alpha \in E_i$ such that $\overline{\lambda} = \alpha^{1/i}$. We continue to denote these elements by $\alpha_{i,1}, \ldots, \alpha_{i,n_i}$. By Lemma 5 we have $\overline{\rho} \neq 0$. Since $p(\rho(q), q) = 0$, it follows from (59) that

$$0 = \lim_{q\to+\infty} \frac{p(\rho(q), q)}{(\overline{\lambda}^q)^p} = \overline{\rho}^p + \sum_{i=1}^{p} \sum_{j=1}^{n_i} (-1)^i (-1)^{\sigma(\alpha_{i,j})} \overline{\rho}^{p-i}. \tag{64}$$

Furthermore, it also follows from (59) that

$$\frac{\partial}{\partial q} p(z, q) = \sum_{i=1}^{p} \sum_{\alpha\in E_i} (-1)^i (-1)^{\sigma(\alpha)} \log \alpha \cdot \alpha^q z^{p-i},$$

and

$$\frac{\partial}{\partial z} p(z, q) = pz^{p-1} + \sum_{i=1}^{p} \sum_{\alpha\in E_i} (-1)^i (-1)^{\sigma(\alpha)} (p - i)\alpha^q z^{p-i-1}.$$

Therefore,

$$\begin{aligned}\lim_{q\to+\infty} \frac{\frac{\partial}{\partial q} p(\rho(q), q)}{(\overline{\lambda}^q)^p} &= \sum_{i=1}^{p} \sum_{j=1}^{n_i} (-1)^i (-1)^{\sigma(\alpha_{i,j})} \log(\overline{\lambda}^i) \overline{\rho}^{p-i} \\ &= \log \overline{\lambda} \sum_{i=1}^{p} \sum_{j=1}^{n_i} (-1)^i (-1)^{\sigma(\alpha_{i,j})} i \overline{\rho}^{p-i},\end{aligned}$$

and

$$\begin{aligned}\lim_{q\to+\infty} \frac{\rho(q) \frac{\partial}{\partial z} p(\rho(q), q)}{(\overline{\lambda}^q)^p} &= p\overline{\rho}^p + \sum_{i=1}^{p} \sum_{j=1}^{n_i} (-1)^i (-1)^{\sigma(\alpha_{i,j})} (p - i) \overline{\rho}^{p-i} \\ &= p\left[\overline{\rho}^p + \sum_{i=1}^{p} \sum_{j=1}^{n_i} (-1)^i (-1)^{\sigma(\alpha_{i,j})} \overline{\rho}^{p-i}\right]\end{aligned}$$

$$-\sum_{i=1}^{p}\sum_{j=1}^{n_i}(-1)^i(-1)^{\sigma(\alpha_{i,j})}i\overline{\rho}^{p-i}$$

$$=-\sum_{i=1}^{p}\sum_{j=1}^{n_i}(-1)^i(-1)^{\sigma(\alpha_{i,j})}i\overline{\rho}^{p-i},$$

using (64) in the last identity. It follows from Lemma 6 that

$$\lim_{q\to+\infty}T'(q)=-\frac{\log\overline{\lambda}\sum_{i=1}^{p}\sum_{j=1}^{n_i}(-1)^i(-1)^{\sigma(\alpha_{i,j})}i\overline{\rho}^{p-i}}{-\sum_{i=1}^{p}\sum_{j=1}^{n_i}(-1)^i(-1)^{\sigma(\alpha_{i,j})}i\overline{\rho}^{p-i}}-P(\eta)=\log\overline{\lambda}-P(\eta).$$

The case when $q\to-\infty$ can be treated in a similar manner. □

We now proceed with the proof of the theorem. Let $\mathcal{E}$ be the entropy spectrum of an LC(2) equivalence class. Given a function $\varphi\in\mathrm{LC}(2)$ that generates the spectrum, in a similar manner to that in the proof of Theorem 3 we may assume, without loss of generality, that $P(\varphi)=0$, and

$$\varphi|C_{12}=\log\alpha_{12},\qquad \varphi|C_{21}=0,\qquad \varphi|C_{31}=0, \tag{65}$$

$$\varphi|C_{13}=\log\alpha_{13},\qquad \varphi|C_{23}=\log\alpha_{23},\qquad \varphi|C_{32}=\log\alpha_{32} \tag{66}$$

with $\alpha_{12}\ge\alpha_{13}\ge\alpha_{23}\alpha_{32}$. Then the matrix $A(\varphi)$ is given by (51). By Theorem 5, if $\alpha_{12}\alpha_{23}\ne\alpha_{13}\alpha_{32}$ then the function ψ with matrix $A(\psi)$ given by (51) has the same entropy spectrum as φ but is not equivalent to φ.

In order to obtain the functions φ and ψ from $\mathcal{E}$, we analyze $T(q)=P(q\varphi)$ or more precisely its derivative $T'(q)$.

Lemma 8 *Let (Σ_A^+,σ) be the topological Markov chain with transition matrix A as in (50), and let $\eta\in\mathrm{LC}(2)$ be a function with $\eta|C_{ij}=\log c_{ij}$ for $i\ne j$ and $P(\eta)=0$. Then the sets*

$$S_1=\{\sqrt{c_{12}c_{21}},\sqrt{c_{13}c_{31}},\sqrt{c_{23}c_{32}}\}, \tag{67}$$

$$S_2=\{\sqrt[3]{c_{12}c_{23}c_{31}},\sqrt[3]{c_{13}c_{32}c_{21}}\} \tag{68}$$

can be determined from $T'(q)$.

Proof of the lemma In view of Lemma 6, it follows from a straightforward computation that

$$\begin{aligned}T'(q)&=\frac{\psi(c_{12}c_{23}c_{31})+\psi(c_{13}c_{32}c_{21})}{3\rho^3(q)-\rho(q)((c_{12}c_{21})^q+(c_{13}c_{31})^q+(c_{23}c_{32})^q)}\\&\quad+\rho(q)\frac{\psi(c_{12}c_{21})+\psi(c_{13}c_{31})+\psi(c_{23}c_{32})}{3\rho^3(q)-\rho(q)((c_{12}c_{21})^q+(c_{13}c_{31})^q+(c_{23}c_{32})^q)},\end{aligned} \tag{69}$$

where $\psi(x)=x^q\log x$. Since $\rho(q)$ is a root of the characteristic polynomial of the matrix $A(q\eta)$ (see (57)), we have

$$\rho^3(q)-\rho(q)((c_{12}c_{21})^q+(c_{13}c_{31})^q+(c_{23}c_{32})^q)=(c_{12}c_{23}c_{31})^q+(c_{13}c_{32}c_{21})^q.$$

Therefore, (69) can be written in the form

$$T'(q) = \frac{\psi(c_{12}c_{23}c_{31}) + \psi(c_{13}c_{32}c_{21})}{2\rho^3(q) + (c_{12}c_{23}c_{31})^q + (c_{13}c_{32}c_{21})^q} + \rho(q)\frac{\psi(c_{12}c_{21}) + \psi(c_{13}c_{31}) + \psi(c_{23}c_{32})}{2\rho^3(q) + (c_{12}c_{23}c_{31})^q + (c_{13}c_{32}c_{21})^q}.$$

By Lemma 7, and (55), and (56), we have

$$\overline{c} = \lim_{q\to+\infty} T'(q) = \log\max(S_1 \cup S_2),$$
$$\underline{c} = \lim_{q\to-\infty} T'(q) = \log\min(S_1 \cup S_2).$$

Since μ_0 is the Markov measure with probability vector $(1/3, 1/3, 1/3)$ and matrix $A/2$, it follows from the identity

$$T'(q) = \int_{\Sigma_A^+} \eta\, d\mu_{q\eta},$$

that

$$\begin{aligned} 6T'(0) &= 2(\log(c_{12}c_{21}) + \log(c_{13}c_{31}) + \log(c_{23}c_{32})) \\ &= 3(\log(c_{12}c_{23}c_{31}) + \log(c_{13}c_{32}c_{21})). \end{aligned} \tag{70}$$

To determine whether $e^{\overline{c}} \in S_2$ or not we study the function

$$\overline{\mathcal{V}}(\alpha) = \lim_{q\to+\infty}\left[\frac{(2\rho^3(q) + e^{3\overline{c}q} + e^{\overline{\kappa}q})T'(q)}{\rho(q)\alpha^q} - \frac{3\overline{c}e^{3\overline{c}q} + \overline{\kappa}e^{\overline{\kappa}q}}{\rho(q)\alpha^q}\right],$$

where $\overline{\kappa} = 6T'(0) - 3\overline{c}$. When $e^{\overline{c}} \in S_2$, by (70) we have $S_2 = \{e^{\overline{c}}, e^{2T'(0)-\overline{c}}\}$, and

$$\overline{\mathcal{V}}(\alpha) = \lim_{q\to+\infty} \frac{\psi(c_{12}c_{21}) + \psi(c_{13}c_{31}) + \psi(c_{23}c_{32})}{\alpha^q}.$$

Thus, there exists $\mathcal{V}^* < 0$ such that

$$\overline{\mathcal{V}}(\alpha) = \begin{cases} -\infty, & 0 \le \alpha < \alpha^*, \\ \mathcal{V}^*, & \alpha = \alpha^*, \\ 0, & \alpha > \alpha^*. \end{cases}$$

Setting

$$\alpha^* := \inf\{\alpha : \overline{\mathcal{V}}(\alpha) = 0\}, \tag{71}$$

we have $\alpha^* \le e^{2\overline{c}}$ and $\overline{\mathcal{V}}(\alpha^*)/\log\alpha_* \in \mathbb{N}$. When $e^{\overline{c}} \notin S_2$, we consider three cases, depending on the sign of $\overline{c} - T'(0)$:

1. We first assume that $\overline{c} > T'(0)$. Let $\overline{n}$ be the number of elements in S_1 that are equal to $e^{\overline{c}}$, and set

$$\overline{\rho} = \lim_{q\to+\infty} \frac{\rho(q)}{e^{\overline{c}q}}$$

(by Lemma 5 the limit exists and is finite). If $p(z,q)$ is the characteristic polynomial of the matrix $A(q\eta)$, we have $p(\rho(q),q)=0$ and thus

$$0=\lim_{q\to+\infty}\frac{p(\rho(q),q)}{e^{3\overline{c}q}}=\overline{\rho}^3-\overline{n\rho}.$$

Hence $\overline{\rho}=\sqrt{\overline{n}}\geq 1$. We can show in a straightforward manner that

$$\overline{\mathcal{V}}(e^{2\overline{c}})=\lim_{q\to+\infty}\left[\frac{2\rho^3(q)+e^{3\overline{c}q}}{\rho(q)e^{2\overline{c}q}}T'(q)-\frac{3\overline{c}e^{3\overline{c}q}}{\rho(q)e^{2\overline{c}q}}\right]=\left(2\overline{\rho}^2+\frac{1}{\overline{\rho}}-\frac{3}{\overline{\rho}}\right)\overline{c}=2\overline{c}\frac{\overline{\rho}^3-1}{\overline{\rho}}.$$

Therefore,

$$\overline{\mathcal{V}}(\alpha)=\begin{cases}\infty, & 0\leq\alpha<e^{2\overline{c}},\\ 2\overline{c}(\overline{\rho}^3-1)/\overline{\rho}, & \alpha=e^{2\overline{c}},\\ 0, & \alpha>e^{2\overline{c}}.\end{cases}$$

Since $\overline{\rho}=\sqrt{\overline{n}}$ we have $\alpha^*=e^{2\overline{c}}$, and $\overline{\mathcal{V}}(\alpha^*)/\log\alpha_*\notin\mathbb{N}$.

2. The case when $\overline{c}=T'(0)$ is analogous to the previous one and thus we omit the details. In this case we can show that

$$\overline{\mathcal{V}}(\alpha)=\begin{cases}\infty, & 0\leq\alpha<e^{2\overline{c}},\\ 2\overline{c}(\overline{\rho}^3-2)/\overline{\rho}, & \alpha=e^{2\overline{c}},\\ 0, & \alpha>e^{2\overline{c}}.\end{cases}$$

We have $\alpha^*=e^{2\overline{c}}$, and $\overline{\mathcal{V}}(\alpha^*)/\log\alpha_*\notin\mathbb{N}$.

3. When $\overline{c}<T'(0)$ we have

$$\overline{\mathcal{V}}(\alpha)=\lim_{q\to+\infty}\frac{e^{(6T'(0)-3\overline{c})q}}{\overline{\rho}e^{\overline{c}q}\alpha^q}[T'(q)-(6T'(0)-3\overline{c})].$$

Thus, $\alpha^*=e^{6T'(0)-4\overline{c}}>e^{2\overline{c}}$, and

$$\overline{\mathcal{V}}(\alpha)=\begin{cases}\infty, & 0\leq\alpha<\alpha^*,\\ (4\overline{c}-6T'(0))/\overline{\rho}, & \alpha=\alpha^*,\\ 0, & \alpha>\alpha^*.\end{cases}$$

To decide whether $e^{\overline{c}}\in S_2$ we proceed as follows:

1. compute α^* (see (71));
2. if $\alpha^*>e^{2\overline{c}}$, then $e^{\overline{c}}\notin S_2$;
3. If $\alpha^*<e^{2\overline{c}}$, then $e^{\overline{c}}\in S_2$;
4. if $\alpha^*=e^{2\overline{c}}$ and $\overline{\mathcal{V}}(\alpha^*)/\log\alpha^*\in\mathbb{N}$, then $e^{\overline{c}}\in S_2$;
5. if $\alpha^*=e^{2\overline{c}}$ and $\overline{\mathcal{V}}(\alpha^*)/\log\alpha^*\notin\mathbb{N}$, then $e^{\overline{c}}\notin S_2$.

A similar analysis to the previous one but for the function

$$\underline{\mathcal{V}}(\alpha)=\lim_{q\to-\infty}\left[\frac{(2\rho^3(q)+e^{3\underline{c}q}+e^{\underline{\kappa}q})T'(q)}{\rho(q)\alpha^q}-\frac{3\underline{c}e^{3\underline{c}q}+\underline{\kappa}e^{\underline{\kappa}q}}{\rho(q)\alpha^q}\right],$$

where $\underline{\kappa}=6T'(0)-3\underline{c}$, allows us to determine whether $e^{\underline{c}}\in S_2$ or not.

We can now determine the sets S_1 and S_2. For S_1 we proceed as follows:

1. If $e^{\overline{c}} \in S_2$ then:

   (a) $S_2 = \{e^{\overline{c}}, e^{2T'(0)-\overline{c}}\}$ and $\sqrt{\alpha^*}$ occurs in S_1 a number of times equal to $\overline{\mathcal{V}}(\alpha^*)/\log\alpha^*$. If $\overline{\mathcal{V}}(\alpha^*)/\log\alpha^* = 3$, then $S_1 = \{\sqrt{\alpha^*}\}$.

   (b) If $\overline{\mathcal{V}}(\alpha^*)/\log\alpha^* \neq 3$, then we analyze the function

$$\overline{\mathcal{V}}_1(\alpha) = \overline{\mathcal{V}}(\alpha) - \lim_{q\to+\infty}\left(\frac{\overline{\mathcal{V}}(\alpha^*)}{\log\alpha^*} \times \frac{\log\alpha^*(\alpha^*)^q}{\alpha^q}\right).$$

   Letting now $\alpha_1^* = \inf\{\alpha : \overline{\mathcal{V}}_1(\alpha) = 0\} > 0$ the number $\sqrt{\alpha_1^*}$ occurs in S_1 a number of times equal to $\overline{\mathcal{V}}_1(\alpha_1^*)/\log\alpha_1^*$. Thus, if

$$\frac{\overline{\mathcal{V}}(\alpha^*)}{\log\alpha^*} + \frac{\overline{\mathcal{V}}_1(\alpha_1^*)}{\log\alpha_1^*} = 3, \tag{72}$$

   then the set S_1 is determined.

   (c) If (72) does not hold then we analyze the function

$$\overline{\mathcal{V}}_2(\alpha) = \overline{\mathcal{V}}_1(\alpha) - \lim_{q\to+\infty}\left(\frac{\overline{\mathcal{V}}_1(\alpha_1^*)}{\log\alpha_1^*} \times \frac{\log\alpha_1^*(\alpha_1^*)^q}{\alpha^q}\right).$$

   Letting $\alpha_2^* = \inf\{\alpha : \overline{\mathcal{V}}_2(\alpha) = 0\} > 0$ the number $\sqrt{\alpha_2^*}$ is the missing element of S_1, i.e., $S_1 = \{\sqrt{\alpha^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}\}$.

2. If $e^{\overline{c}} \notin S_2$ and $e^{\underline{c}} \in S_2$, then we proceed as in the previous case, with $\underline{c}$ and $\underline{\mathcal{V}}$ playing respectively the roles of $\overline{c}$ and $\overline{\mathcal{V}}$.
3. If $e^{\overline{c}} \notin S_2$ and $e^{\underline{c}} \notin S_2$, then

$$S_1 = \{e^{\overline{c}}, e^{3T'(0)-\overline{c}-\underline{c}}, e^{\underline{c}}\}.$$

To determine S_2 we analyze the function

$$\mathcal{W}(\alpha) = \lim_{q\to+\infty}\left[\frac{3\rho^3(q) - \rho(q)(e^{2\overline{c}q} + e^{\gamma q} + e^{2\underline{c}q})}{\alpha^q}T'(q) - \rho(q)\frac{2\overline{c}e^{2\overline{c}q} + \gamma e^{\gamma q} + 2\underline{c}e^{2\underline{c}q}}{\alpha^q}\right],$$

where $\gamma = 6T'(0) - 2\overline{c} - 2\underline{c}$. By (69) we have

$$\mathcal{W}(\alpha) = \lim_{q\to+\infty}\frac{\log(c_{12}c_{23}c_{31})(c_{12}c_{23}c_{31})^q + \log(c_{13}c_{32}c_{21})(c_{13}c_{32}c_{21})^q}{\alpha^q}.$$

Letting $\alpha^* := \inf\{\alpha : \mathcal{W}(\alpha) = 0\} > 0$, we obtain

$$S_2 = \{\sqrt[3]{\alpha^*}, \sqrt[3]{e^{2T'(0)-\log\alpha^*}}\}.$$

This yields the desired result. □

We now complete the proof of the theorem. In view of (65) and (66), it follows from Lemma 8 that from the function T, and thus from the spectrum $\mathcal{E}$, we can determine the sets S_1 and S_2 in (67) and (68). Therefore, we can also determine the sets

$$T_1 = \{\alpha_{12}, \alpha_{13}, \alpha_{23}\alpha_{32}\} \quad \text{and} \quad T_2 = \{\alpha_{12}\alpha_{23}, \alpha_{13}\alpha_{32}\}.$$

Note that although we know these sets, we are not able to say which element of T_2 corresponds to $\alpha_{12}\alpha_{23}$.

Since $\alpha_{12} \geq \alpha_{13} \geq \alpha_{23}\alpha_{32}$, letting $\beta_1 = \max T_1$, $\beta_3 = \min T_1$, and β_2 the remaining element of T_1, the matrix associated to the function φ is of the form

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 \\ 1 & 0 & \beta_3/x \\ 1 & x & 0 \end{pmatrix} \quad \text{for some } x > 0.$$

Letting $T_2 = \{\gamma_1, \gamma_2\}$, we find x from $\beta_2 x = \gamma_1$ or $\beta_2 x = \gamma_2$. We thus obtain two possible matrices, namely $A(\varphi)$ and $A(\psi)$ in (51).

The case when we recover a unique equivalence class occurs when the elements of the set T_2 are equal, i.e., when $\alpha_{12}\alpha_{23} = \alpha_{13}\alpha_{32}$. $\square$

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